IVICA MARTINOVIĆ

Boscovich on the Problem of *Generatio Velocitatis*: Genesis and Methodological Implications

Estratto da

R.J. Boscovich
Vita e attività scientifica
His life and scientific work

A cura di Piers Bursill-Hall

Roma, Istituto della Enciclopedia Italiana, 1993

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Immediate Production of Velocity: Boscovich's Key to the Interpretation of Phenomena

The dispute over *vires vivae* offered Boscovich an opportunity to re-examine the very foundations of dynamics in his treatise *De viribus vivis* [1]. The peculiarity of Boscovich's approach is recognizable in his understanding of the relationship between force and velocity. Boscovich started his exposition of dynamics with the assertion that all natural phenomena depend on the force of inertia and momentary or permanent actions of the *potentiae* or «dead» forces, «live» forces being thus completely unnecessary [2]. The force of inertia is a certain determination (*determinatio*) of matter by which it remains in the state of rest or uniform motion in the direction it has been originally set in, until it is forced by another *potentia* to change this state (cf. n. 10 *infra*). The *potentiae* (potentials) are causes which change the state of a body by their actions, and the examples of such causes, quoted by Boscovich, are: the impenetrability (*impenetrabilitas*) in collision between bodies, the gravity (*gravitas*) towards the centre or other body, the elastic force (*vis elastica*), the cause of adhesion (*causa adhaesionis*), the resistence to compression (*causa obsistens compressioni*), and other such causes, if they occur (cf. n. 13 *infra*).

Boscovich noticed that the force of inertia and potentials relate to velocity, though in different ways: «With the force of inertia the bodies, if they have no velocity, rest, but if they have some velocity, they retain the same one until some potential (*potentia*) generates a new velocity» [3]. Therefore, change of velocity does

[1] R.J. Boscovich [1745a], pp. 10-13, nn. 9-14. Cf. B. Truhelka, *Rudžer Josip Bošković. Ulomci biografije*, in *Građa* [1950-1957], vol. I, pp. 91-221, 125-140; P. Costabel [1961], pp. 3-12; P. Costabel, *Le rôle du continu dans la genèse de la pensée de R. Bošcović en mecanique*, in *Actes* [1962], pp. 107-114; Ž. Marković [1968-69], vol. I, pp. 177-187; I. Martinović [1986], pp. 3-22: 6-7; L. Indorato, P. Nastasi [1987], 1-2, pp. 59-80.

[2] «Phaenomena omnia ita pendere a vi inertiae, et momentaneis, ac perpetuo pereuntibus potentiarum actionibus, sive viribus mortuis, ut vires vivae sint prorsus superfluae» (R.J. Boscovich [1745a], n. 9, p. 10).

[3] «Ea vi inertiae corpora si nullam habent velocitatem, quiescunt; si habent aliquam, eandem retinent, donec nova ab aliqua potentia generetur» (R.J. Boscovich [1745a], n. 10, p. 11).

not occur through the force of inertia but only by the action of a potential. Velocity generation (*generatio velocitatis*) should just be connected with the process of the acting *potentia* (potential).

Boscovich himself, especially in his later works, pointed out that velocity generation is the inspiring idea of his dynamics and that it is a new approach, obtainable only from the view-point of his understanding of continuity and infinity. I shall quote two examples. When he analysed de Maupertuis's work *Essay de Cosmologie* (1732) Boscovich noticed that one of the two facts de Maupertuis warned the scientific community about is: it is unknown to us how velocity is generated and whether continuity is interrupted in this way. This remark made Boscovich want to show in the treatise *De continuitatis lege* that velocity appears and disappears always according to the law of continuity [4]. In the postscript to Giovanni Arnolfini of 3 February 1770 he mentioned that one of the ideas Antonio Lecchi was advised to insert in *Idrostatica* (1765) was «*l'origine e la genesi delle velocità*» [5].

Re-examining the process of the action of force and generation of velocity respectively, Boscovich, as he himself confessed, was inspired by the scholastic distinction *in actu primo - in actu secundo* [6], but with an essential difference relating to Leibnizian dynamic. At the end of the 17th century Leibniz used the Aristotelo-scholastic distinction *materia prima - materia secunda* when he treated forces: absolute and respective, total and partial, relative and directive. The distinction between «dead» and «live» forces belongs to the same ideational origin [7]. Boscovich considered it more appropriate to apply this distinction to the notion of velocity.

The velocity *in actu secundo* is a certain relation between the distance covered and the time taken. Boscovich states that this idea does not include anything else except: time, distance and a relation according to which the velocity is called greater if during uniform motion a greater distance is covered within the same time or less time is needed for the same distance. Therefore, in the notion of velocity in the second act one should recognize the velocity of uniform motion in a straight line $c=s/t$. The velocity in the first act (*in actu primo*) presents the determination itself (*determinatio*) which a body has in comparison to the velocity in the second act, i.e. the determination to cross a certain distance in the given time. Such velocity is to be recognized just as the velocity of inertia. The notion of *determinatio*, by which the ideas of velocity in the first act and the force of inertia are explained, is untranslatable into the language of mathematics. It belongs to the field of philosophical considerations. In later works by Boscovich the velocity in the second act is also called the actual velocity, and the velocity in the first act the potential one.

[4] «Et quidem quod ad modum pertinet, quo velocitas generatur, ostendemus inferius, eam juxta continuitatis legem generari semper, ac destrui» (R.J. Boscovich [1754a], p. 46, n. 105); see also p. 63, n. 138.

[5] R.J. Boscovich [1963a], pp. 1-79: 27.

[6] R.J. Boscovich [1745a], p. 11, n. 11.

[7] G.W. Leibniz, *Specimen dinamicum*, in *Hauptschriften zur Grundlegung der Philosophie*, ed. by E. Cassirer, Hamburg, 1966, vol. I, pp. 256-272: 259-264.

Then, Boscovich establishes that the *potentiae* (potentials), denoting any kind of cause, *in the same way* generate the velocity in the second act and determine the same velocity in the first act [8]. In this way, he announces the general solution, i.e. the solution that can be applied to all phenomena of motion, not using the notion of the *vis viva* (live force) in any way. Further, he verifies that solution in a number of cases: uniform motion in a straight line (cf. n. 20 *infra*), free fall in vacuum (cf. n. 22 *infra*), the collision of bodies (cf. nn. 24, 25, 34 *infra*), the action of the magnetic force (cf. n. 24 *infra*), impenetrability (cf. 30 *infra*), penetration into soft bodies (cf. nn. 31, 32, 35 *infra*), without feeling the need for experimental confirmation of his conceptions. He does not carry out *new* experiments, but he tries to reinterpret already known experimental results in terms of his ideas. At the end of his verification he concludes:

> Having elaborated in this way all kinds of phenomena, we have shown there is nowhere necessity for live forces, but that all motion phenomena of every kind can be explained from the simplest principles by the immediate production of velocity exerted so that after the action of a potentia nothing remains in the bodies concerned except the opposite determination of the force of inertia by which determination the very force of inertia perseveres [9].

Boscovich's general interpretation of natural phenomena is based on the immediate production of velocity and can be explained by the simplest principles. What kind of velocity production does Boscovich imagine, and which are the simplest principles he uses in dynamics? I shall show in detail how the solution appears and matures in Boscovich's thought. Six different sources from the period 1745-1770 testify to the genesis of the problem of *generatio velocitatis* in Boscovich's scientific work:

(a) Boscovich's letter to Federico Sanvitali, 20 March 1745;
(b) The treatise *De viribus vivis* (1745), nn. 15-17, 22-23;
(c) The treatise *De continuitatis lege* (1754), nn. 105, 138, 164;
(d) The supplement *De compositione motuum, ac virium agentium secundum eandem rectam* to the poem *Recentioris philosophiae a Benedicto Stay ... versibus traditae libri X*, Vol. I, (1755), pp. 374-376, nn. 149-165;
(e) *Theòria philosophiae naturalis* (1758), nn. 175-176, annotations (l) and (m); n. 190;
(f) Boscovich's letter to Giovanni Attilio Arnolfini, 2 February 1770.

The Infinitesimal Analysis of Free Fall

Among these sources the first two are privileged since by Boscovich's own admission they express the immediate stream of his thoughts. The letter to Federico

[8] R.J. Boscovich [1745a], p. 12, n. 14.

[9] «Hoc pacto per omnia phaenomenorum genera excurrendo, ostendimus, nusquam viribus vivis opus esse: sed ex simplicissimis principiis omnia omnium generum phaenomena motuum explicari posse per immediatam celeritatis productionem factam actionibus potentiarum nihil post se relinquentibus in ipsis corporibus, praeter diversam determinationem vis inertiae, cum ipsa vi inertiae perseverantem» (R.J. Boscovich [1745a], p. 28, n. 36).

Sanvitali is a preserved fragment of Boscovich's correspondence with Federico Sanvitali, professor of mathematics (1743-1759) at the Collegium Brixiense in Brescia and with Vincenzo Riccati, professor of mathematics (1739-1773) at the Collegium Bononiense in Bologna [10]. It is kept in Camillo Ugoni's transcription in the Archivio dell'Osservatorio astronomico di Brera [11]. The letter to Sanvitali ends with this note:

> Questo è quello che mi è nato sotto la penna. Ieri sera ebbi la sua, onde ho dovuto scrivere in somma fretta, e neppure ho tempo di rileggere la lettera, che è riuscita tanto lunga per non aver avuto tempo da farla più corta [12].

According to his later testimony, Boscovich worked out the treatise *De viribus vivis* «in the same order as it came to his mind the first time» [13]. Should it be pointed out that a crystallization of the original thought, whatever the author subsequently reflected on it, is of extreme importance for an authentic interpretation of the genesis of Boscovich's dynamics? Nevertheless, later testimony of a scientist should be critically examined.

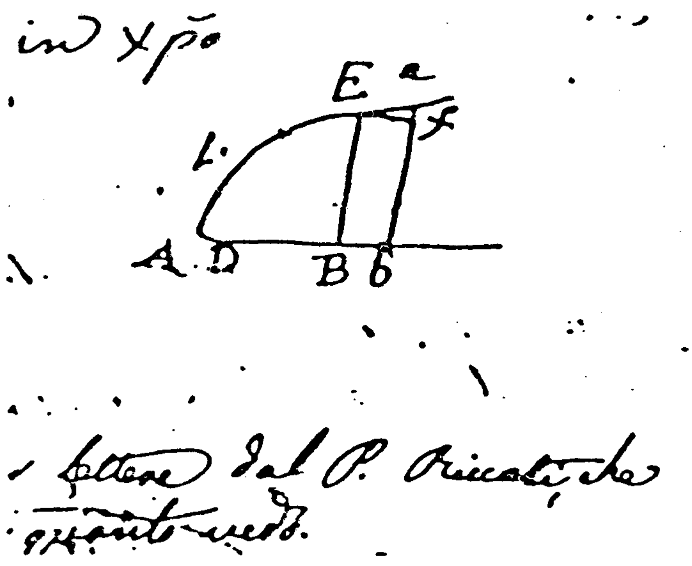

Fig. 1 - Parabola AE: infinitesimal analysis and dynamic explanation. From Boscovich's letter to F. Sanvitali, 20 March 1745. Courtesy of the Archivio dell'Osservatorio astronomico di Brera.

[10] R.J. Boscovich [1887-88], pp. 250-255:250: «Manco male, che non sono solo a non ricever lettere del P. Riccati, che non trova la via di rispondere a niuno per quanto vedo». Also published in R.J. Boscovich [1938], pp. 21-141: 26-30. For a detailed treatment of the professors of mathematics at the Collegium Brixiense and Collegium Bononiense see K.A.F. Fischer [1983], pp. 52-92: 80.

[11] A. Mandrino, G. Tagliaferri, P. Tucci [1986], p. 7, n. 201.

[12] R.J. Boscovich [1887-88], p. 255.

[13] «Pleraque ex iis, quae dicturus sum, duobus ad hinc annis partim exposui, partim innui in dissertatione de Viribus Vivis edita primum hic Romae anno 1745, [...], minus digesta illa quidem, et eodem ordine, quo primum in mentem venerant ennunciata» (R.J. Boscovich [1757b], pp. 129-258: 136).

Not having received Vincenzo Riccati's letter [14], Boscovich began the letter to Sanvitali with a commentary on par. 273 of the second volume of Newton's *Principia*, edited by Le Seur and Jacquier in 1740. He considered the parabola AE with the axis AB (see Figure 1) as a purely geometrical problem and applied the procedures of differentiation and integration to it [15].

If $\mathrm{AD}=a$, $\mathrm{AB}=a+x$, $\mathrm{BE}=y$, then the connection between variables AB, BE is expressed by the equation of a parabola:

$$p\ (a+x)=y^2,$$

$$ap+px=yy.$$

If the increments of these quantities are introduced:

$$\mathrm{Bb}=\mathrm{d}x,\ \ \mathrm{Ab}=a+x+\mathrm{d}x,\ \ \mathrm{be}=y+\mathrm{d}y,$$

then the connection between the quantities Ab, be will be expressed according to the nature of the curve (*per la natura della curva*):

$$p(a+x+\mathrm{d}x)=(y+\mathrm{d}y)^2,$$

$$ap+px+p\mathrm{d}x=yy+2y\mathrm{d}y+\mathrm{d}y^2.$$

The method of differentiation is based on the principle that the equality of variables causes the equality of their increments. Therefore,

$$p\mathrm{d}x=2y\mathrm{d}y+\mathrm{d}y^2,$$

i.e. disregarding $\mathrm{d}y^2$ as an infinitely small quantity compared to $\mathrm{d}y$, as well as $\mathrm{d}y$ as an infinitely small quantity compared to $2y$:

$$p\mathrm{d}x=2y\mathrm{d}y.$$

The method of integration is based on the principle that equality of the increments implies equality of the variables. If

$$2y\mathrm{d}y=2r\mathrm{d}x,$$

[14] It was known to Boscovich that Vincenzo Riccati, an Italian mathematician and a Jesuit, accepted Leibnizian views of living forces, which follows from the historical account of the vis viva controversy in R.J. Boscovich [1745a], p. 5, n. 4. See also R.J. Boscovich, *De variis virium activarum generibus, et earum effectu*, a supplement to [1755a], vol. I, p. 372, n. 141, where Boscovich examines Riccati's view expressed «in elegantissimis Dialogis suis de Viribus vivis».

[15] R.J. Boscovich [1887-88], pp. 251-253.

we obtain by integration:

$$y^2 = 2rx + q,$$

where q is an arbitrary quantity.

The second step is the application of this geometrical problem to mechanics. But this time it is not an application to kinematic relationships as he treated them in the treatise *De motu corporum projectorum in spatio non resistente* (1740) when he calculated the distance covered by uniformly accelerated motion with the help of Newton's method of prime and ultimate ratios [16]. It is an application to dynamic relationships, specifically to the dynamic analysis of free fall, as follows from the initial supposition: «*scenda un grave verso O colla direzione DO con una forza costante*» [17]. According to the preserved copy of Boscovich's letter it comes out that Boscovich measures the path on the abscissa, not the time, as should be done. Although at the end of this letter he admits that he made some mistakes, Boscovich establishes the relationships between the force, velocity, distance and time in the free fall in the following way. The distance ds, crossed in the time dt, is proportional to time interval dt and the velocity u:

$$u\mathrm{d}t = \mathrm{d}s.$$

Supposing a constant force, the velocity, or rather the change of the velocity, du is proportional to the time dt in which it is generated and to the force f which generates it:

$$f\mathrm{d}t = \mathrm{d}u.$$

Hence

$$\mathrm{d}t = \frac{\mathrm{d}s}{u}, \quad \mathrm{d}t = \frac{\mathrm{d}u}{f},$$

$$\frac{\mathrm{d}s}{u} = \frac{\mathrm{d}u}{f}$$

$$f\mathrm{d}s = u\mathrm{d}u,$$

$$2f\mathrm{d}s = 2u\mathrm{d}u.$$

This is the differential equation that gives, upon «passing to the integral», as Boscovich previously showed:

$$2fs + q = uu,$$

16 Cf. I. Martinović [1988], 3-4, pp. 57-71: 64-67.

17 R.J. Boscovich [1887-88], p. 253, leading to the dynamic analysis of free fall on pp. 253-254.

where q is determined from the nature of the problem (*della stessa natura del problema*). If a body falls freely, the velocity at the beginning of the path is equal to zero, hence $q = 0$; if a body is started at the velocity b, then $q = b^2$. Here Boscovich supports a wrong opinion that adding of the constant does not change the nature of a curve in these examples, but in many other cases does.

Velocity Generation in the General Case: Primary Conception

The third step in the genesis of the problem of the generation of velocity is the generalization of the dynamical relationships under consideration in order to explain velocity generation in the general case. This step was taken by Boscovich in the treatise *De viribus vivis*, but in a rather undeveloped way (*minus digesta* as he himself says) [18], so it should be reconstructed from the text itself. In contrast to the previous two stages Boscovich does not use the techniques of differentiation and integration directly, but he does use the infinitesimals of dynamic quantities and refers to the law of infinitesimals. What are the reasons for such a choice of method?

Obviously, it is a case of social reasons. The treatise is intended for the alumni of the Collegium Romanum and for the selected audience traditionally invited to the solemn annual exercise. But, the reason also lies in a new expressive technique which Boscovich will use in further steps, whenever he questions the problem of *generatio velocitatis*: the dynamic diagram with a fourfold purpose (see Figures 2, 3, 4) [19]. Pierre Costabel showed that it is s'Gravesande's figure, perceived originally by Boscovich [20]. Boscovich now introduces the geometrical presentation and, although he assumes the fundamentals of infinitesimal calculus in its interpretation, in the exposition of the problem, without using differential equations, he gives a distinctive advantage to the geometrical approach in relation to the infinitesimal method, which completely corresponds with Boscovich's choice of mathematical method from 1740 [21].

Approaching the velocity generation problem in *De viribus vivis* Boscovich introduces an intermediary between a *potentia* (potential) or cause of motion and velocity. It is pressure (*pressio*). It is produced by a *potentia* in some time moments, where it does not convert to velocity by multiplying a pressure by itself some number of times, that is by operation of power, but by conducting through continuous time (*sed solo ductu per tempus continuum*). Here Boscovich explains the process concerned: «just as a line does not become an area by multiplying by itself any

[18] See note 13.

[19] See the same diagram in: R.J. Boscovich [1745a], fig. 3; [1755a], vol. I, tab. I, fig. 2; Boscovich's own drawing in his letter to Giovanni Arnfolini, 2 February 1770, in R.J. Boscovich [1963a], p. 13.

[20] P. Costabel, *Le rôle du continu dans la genèse de la pensée de R. Bošcović en mecanique*, in *Actes* [1962], p. 108.

[21] R.J. Boscovich [1740b], scholium on pp. 7-8. Cf. I. Martinović [1988], p. 67.

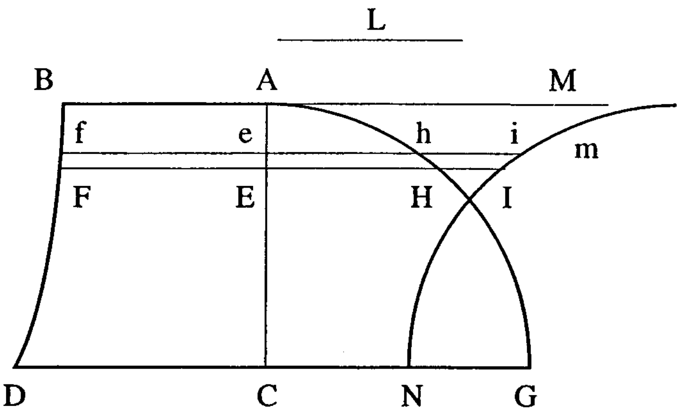

Fig. 2 - In the treatise *De viribus vivis* [1745a], Fig. 3. Courtesy of the Historical Archives (Historijski arhiv), Dubrovnik.

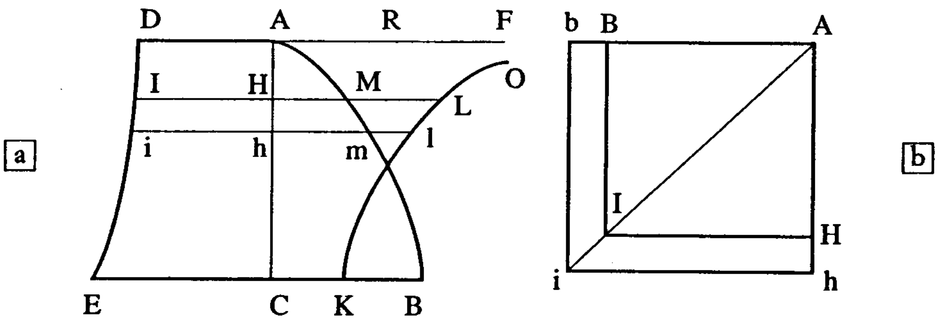

Fig. 3 - In the supplement *De compositione motuum* to Stay's poem [1755a], Fig. 2 and 3. Courtesy of the Library of the Friars Minor (Knjižnica Male braće), Dubrovnik.

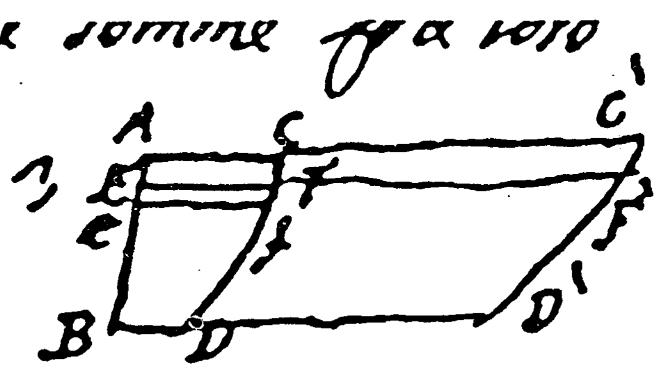

Fig. 4 - Boscovich's own drawing in his letter to G. Arnolfini 2 February 1770, published by Gino Arrighi, cf. R.J. Boscovich [1963a].

number of times, but by continuous conducting along another line» [22]. It is the procedure of defining the definite integral and, consequently, the confirmation

[22] R.J. Boscovich [1745a], p. 13, n. 15: «prorsus ut linea nulla sui multiplicatione evadit superficies, sed continuo ductu per aliam lineam». Cf. P. Costabel, *Le rôle du continu...*, cit., in *Actes* [1962], p. 110.

that Boscovich used the geometrical relationship which Leibniz had used to describe the relation between dead and live force to describe the relation between pressure and velocity. The algebraic equivalent of conducting the pressure through continuous time is a product of pressure with the differential of time. This correlation between the geometrical and algebraic expressions is pointed out by the Latin notion *ductus, productus*. In the case of the geometrical idea it should be translated as *led, conducted*, when expressing the infinitesimal idea of flux, and in the case of relations among algebraic quantities it should be translated as *multiplied*, then denoting very often multiplication by the differential of time or distance. If pressure acting as a link between force and velocity is recognized as the acceleration, the algebraic equivalent is the well-known formula:

$$\mathrm{d}v = a \, \mathrm{d}t.$$

Potential, force, pressure, velocity, time and distance are the basic quantities of Boscovich's dynamics. Therefore, Boscovich's considerations concerning the relationships among these fundamental quantities in the general case are extracted here from the text of the treatise *De viribus vivis*.

> 16. Finally, if the small path Ee is considered indefinitely small, then, because of the difference between the lines ef, EF, the small area FEef is considered indefinitely small as a rectangle. And since the difference between the two pressures in the indefinitely small time is indefinitely small, the non-uniformly accelerated motion in the same small time is considered uniformly accelerated. As a result, if the segments AE of the straight line AC express times, and the straight line EF expresses the same pressures, then the area BAEF will best express the said velocity, in which the pressure EF, continued in time AE, is being transformed. In all other types of quantities, the velocities could also be represented by lines, still arbitrarily assuming a certain line which expresses a certain velocity [23].
> 22. If, however, such velocities [from n. 16] are expressed by ordinates EH of a certain line AHG, then small distances, produced by velocity EH in the small time Ee, will be as small areas EHhe, and the whole paths which correspond to times AE, AC will be as areas AEH, ACG [24].
> 23. And if AE no longer shows times, but rather distances, and EF shows forces which in singular equal small times generate the velocities that are proportional to them, then the small area FEef does not show the velocity produced on the

[23] «Ut demum si spatiolum Ee concipiatur indefinite parvum, ob differentiam rectarum ef, EF indefinite parvam concipitur areola FEef tanquam rectangulum. Ita ob binarum pressionum differentiam tempusculo indefinite parvo indefinite parvam, motus etiam difformiter acceleratus tempusculo illo ipso concipitur tanquam uniformiter acceleratus. Quamobrem si segmenta AE rectae AC exprimant tempora: recta autem EF exprimat pressiones ipsas; optime per planum BAEF exprimetur ipsa celeritas, in quam abit pressio EF continuata per tempus AE. Quanquam ut omne aliud quantitatum genus, velocitates quoque per lineas exprimi poterunt, assumpta ad arbitrium una aliqua linea, quae unam aliquam velocitatem exponat» (R.J. Boscovich [1745a], p. 14, n. 16).

[24] «Si autem ejusmodi velocitates exprimantur per ordinatas EH ad lineam quandam AHG, spatiola tempusculo Ee producta a velocitate EH erunt, ut areolae EHhE, et tota spatia respondentia temporibus AE, AC, ut areae AEH, ACG» (*ibidem*, pp. 17-18, n. 22).

path Ee, because if the small distance is covered more quickly, the same force will produce a smaller velocity. The velocity, however, will be produced in composite ratio from the directly proportional force FE and a small time, which small time is in direct ratio to the small path Ee and in inverse ratio to the whole velocity. The increment of velocity will be in direct ratio to FE and Ee, and in inverse ratio to the whole velocity. As a result, the product of this velocity and its increment will be as a small area FEef. Hence based on the law of infinitesimals it follows that the square of the velocity of the body, which is descending from the point A from rest, is as a small area BAEF. And the same is thought about the decrement that leads to the contrary velocity. If the motion starts in A towards C with velocity that is expressed by BACD, and if the forces act in the opposite direction, then the squares of remaining velocities will be as CEFD, and the whole motion will end in C. And if EI is the ordinate of a certain continuous line NIM which is in inverse ratio to the velocity EH, that is, if it relates to a given line L, as that line relates towards EH, then the small area EIie will be in direct ratio to the small path Ee and will be in inverse ratio to the velocity EH, which is to say that it will be as a small time. Hence the whole time in which AE is crossed will be as the whole area MAEIm [25].

The quoted passages contain four interpretations of the same diagram (see Figure 2).

(a) *Dependence of pressure on time p(t)* (see Figure 5a). The coordinate system in which time intervals (not distances as was erroneously done in the second step), correspond to abscissas and pressures correspond to ordinates can be used to represent the velocity as the area of a portion of the plane. The concept of an indefinitely small quantity (*indefinitum parvum*) is applied to small distance (*spatiolum*), time (*tempusculum*), and area (*areola*). The idealization that the non-uniformly accelerated motion in the infinitely small time interval can be considered uniformly accelerated is accepted. Therefore, having in mind the identification of pressure and acceleration $p = a$, the velocity generated in a moment dt is equal to the area of the small rectangle FEef:

$$\mathrm{d}v = p\ \mathrm{d}t = a\ \mathrm{d}t.$$

[25] «Quod si jam AE exprimat non tempora, sed spatia, et EF vires, quae singulis aequalibus tempusculis generent velocitates sibi proportionales; jam areola FEef non exprimet velocitatem genitam spatiolo Ee, quia quo celerius id spatiolum percurritur, eo minorem velocitatem generabit vis eadem. Erit autem celeritas producta in ratione composita ex directa vis FE et tempusculi, quod tempusculum cum sit ut spatiolum Ee directe, et velocitas tota inverse; erit velocitatis incrementum directe ut FE, et Ee, et reciproce ut tota velocitas, ac proinde productum ex velocitate in suum incrementum, erit ut areola FEef. Inde autem ex infinitesimorum lege colligitur, fore quadratum celeritatis corporis ex A descendentis ex quiete, ut est areola BAEF: et cum de decremento in velocitatem contrariam ducto idem discursus sit; si motus incipiat in A ver[s]us C cum velocitate, quam exprimat BACD, et vires contraria directione agant; erunt residuarum velocitatum quadrata ut CEFD, et motus in C extinguetur totus. Si autem sit EI ordinata ad lineam quandam continuam NIM reciproca velocitatis EH; quae nimirum sit ad datam quandam rectam L, ut haec ad EH; erit areola EIie directe ut spatiolum Ee, et reciproce ut velocitas EH, nimirum ut tempusculum; ac proinde totum tempus, quo percurritur AE, erit, ut tota area MAEIm» (*Ibidem*, pp. 18-19, n. 23).

Four interpretations of the same dynamic diagram:
Extracts from figures 2 and 3 published by Boscovich in 1745 and 1755.

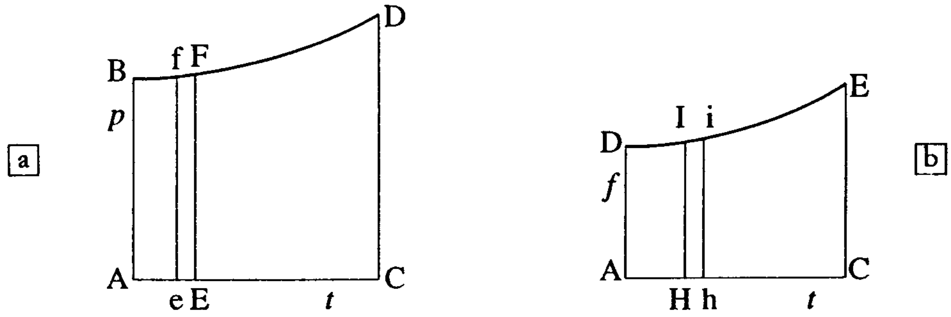

Fig. 5 - Dependence $p(t)$ and $f(t)$ respectively. Area represents velocity.

Thus, the velocity generated in the time interval AC is equal to the area BAEF expressed in terms of the definite integral:

$$v_{AC} = \int_{A}^{C} p \; dt = \int_{A}^{C} a \; dt.$$

(b) *Dependence of velocity on time v(t)* (see Figure 6a). Time is expressed by the abscissa, while velocity, which was represented in the previous coordinate system by an area, is expressed by the ordinate. The velocity is considered constant over an infinitely small time, thus, the small distance covered within that time is represented by a small rectangle:

$$ds = v \; dt.$$

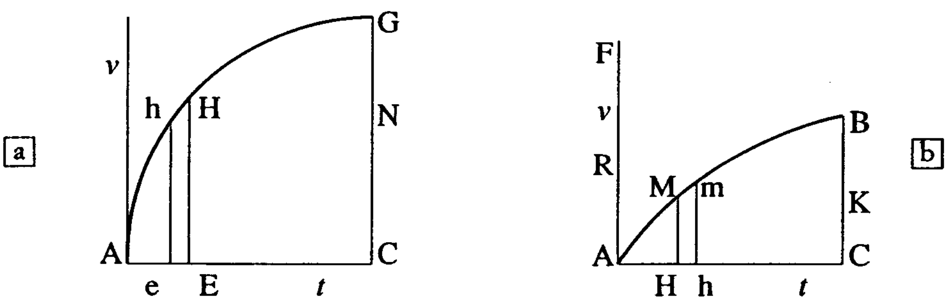

Fig. 6 - Dependence $v(t)$. Area represents distance.

Here, Boscovich uses the language of proportions, i.e. the language of the fifth book of Euclid's *Elements*. In that sense he also establishes the distances in time AE, AC with the help of the proportion of corresponding areas:

$$s_{AE} : s_{AC} = \int_{A}^{E} v \; dt : \int_{A}^{C} v \; dt.$$

(c) *Dependence of force on distance f(s)* (see Figure 7a). The terminology is no longer uniform. Boscovich exchanges the notion of pressure with the notion of force (*pressio* → *vis*). The question of what is represented by a small area, Ee and whose height is the constant force EF along that path, no longer has an answer that is algebraically expressed by simple multiplication. The velocity change is properly called the velocity increment (*velocitatis incrementum*), instead of the velocity, as it was called in the second step. Then, according to Boscovich, we have:

$$dv = f\ dt = f\frac{ds}{v}\ ;$$

$$v\ dv = f\ ds = P_{FEef}.$$

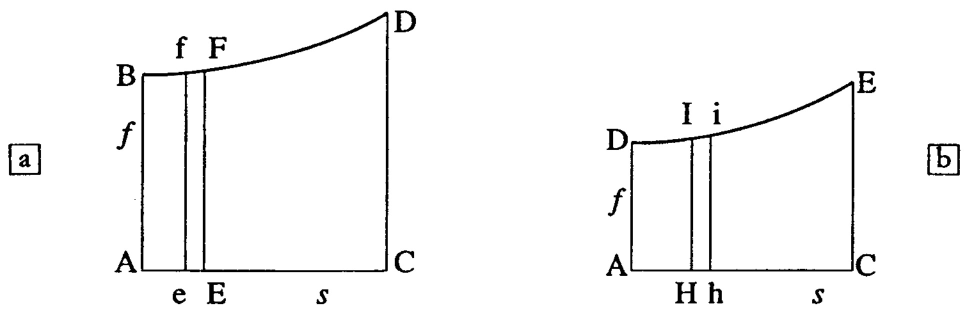

Fig. 7 - Dependence *f*(*s*). Area represents the square of the velocity.

If $v_A = 0$, then:

$$v_E^2 = \int_A^E f\ ds,\ v_C^2 = \int_A^C f\ ds,\ v_C^2 - v_E^2 = \int_E^C f\ ds$$

Boscovich concludes from the law of infinitesimals. This is obviously the same procedure which is called «the transition to the integral» in his letter to Federico Sanvitali.

(d) *Dependence of the reciprocal value of velocity on distance* $\frac{1}{v}$ (*s*) (see Figure 8a).

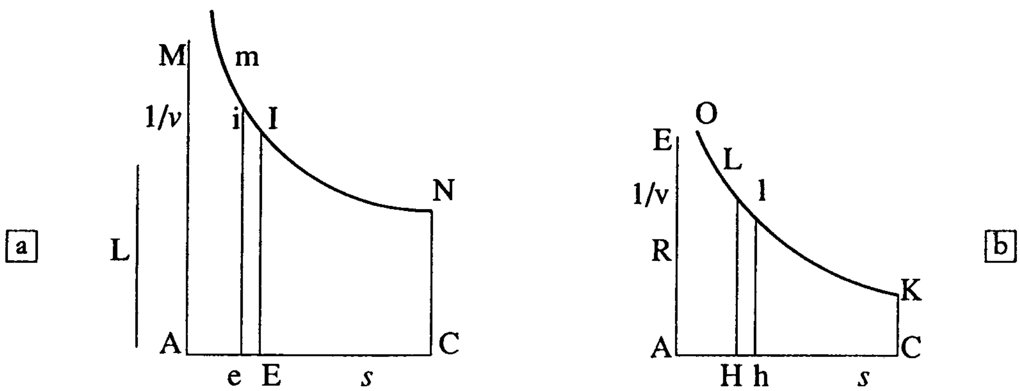

Fig. 8 - Dependence $\frac{1}{v}$ (*s*). Area represents time.

The coordinate system in which distances correspond to abscissas and the reciprocal values of velocity correspond to ordinates is used to represent time as an area. While determining the reciprocal value of the velocity EI, Boscovich uses the velocity representation EH in the second coordinate system and the proportion

$$\text{EI:L} = \text{L:EH},$$

where the segment L, drawn above the basic diagram (see Figure 2), obviously represents the unit of length. He explicitly assumes that the line $\frac{1}{v}(s)$ is continuous, something he did not assume for the line $v(t)$ in the second coordinate system. Moreover, while exploring the first coordinate system, he mentions that the velocity can be assumed arbitrarily. The continuity of velocity is tacitly understood there, taking into consideration the infinitesimal explanation of the generation of velocity. Boscovich, however, does not say anything about what happens at the beginning of motion, namely for $v = 0$, although the ordinate is an asymptote of the line $\frac{1}{v}(s)$ according to the diagram.

The infinitely small time (*tempusculum*) is recognized in this coordinate system as the small rectangle EIie, the base of which is the small path Ee, and the height the reciprocal value of the velocity EI:

$$\mathrm{d}t = \frac{1}{v}\,\mathrm{d}s.$$

Then the time interval spent on the path AE is obtained as the area of MAEIm:

$$t_{\mathrm{AE}} = \int_{\mathrm{A}}^{\mathrm{E}} \frac{1}{v}\,ds.$$

In order to determine that time interval, Boscovich uses again the language of proportions (*esse ut*). In addition, the area of MAEIm, which represents it in a physical sense, is considered to be finite. This is an example of how Boscovich, on the basis of a physical event, tacitly states a mathematical result.

Among the vague attitudes in Boscovich's exposition in this stage one can distinguish the following:
(a) uneven terminology for force and its immediate effect respectively;
(b) arbitrary character of the line expressing velocity;
(c) the definite integral, sometimes expressed by the area under the curve, and sometimes expressed by the ratios of such areas;
(d) tacit assertion of finiteness of the area bounded by the asymptotic arc, stated on the grounds of the physical event instead of the mathematical proof.

Boscovich's objection from 1757, that the exposition in the treatise *De viribus vivis* has not been worked out in a satisfactory manner, refers with good reason to such an explanation of the generation of velocity.

Boscovich's next step in research on the velocity generation problem is characterized by a clear choice of mathematical method and systematic exposition of the problem. While in the treatise *De viribus vivis* Boscovich only touched on the composition of motion within his considerations of basic dynamic relationships [26], in his supplement *De compositione motuum* to the first book of Benedict Stay's poem *Recentioris philosophiae ... versibus traditae libri X* (1755) (nn. 149-165) he just studied this subject to discuss mutual connections between the four basic quantities: the active force of acceleration (*vis activa acceleratrix*, u), the time in which it acts (*tempus, t*), the velocity generated by it (*celeritas*, c), and the distance covered (*spatium, s*). To establish their relationship meant for Boscovich, in this case, to expose the law «on account of which the acceleration and deceleration is generated for various laws of forces» (*qua lege fiat acceleratio, ac retardatio, pro diversa virium lege*) [27]. According to Boscovich's additional and repeated explanation, the law of forces is expressed depending on distance or time (*per spatia, vel tempora*). For that purpose, in accordance with the eighteenth-century concept of a function, Boscovich used scales of forces and velocities (*virium, ac celeritatum scalae*) [28]. Two different approaches are feasible: by means of differential equations (*per algebraicas formulas differentiales*) (cf. n. 150 *infra*) or by means of geometrical constructions (*per Geometriam*) (cf. nn. 151-165 *infra*).

The solution by means of differential equations is the same one Boscovich used in the dynamic analysis of free fall in 1745 and it is reduced to the differential equation:

$$u\ \mathrm{d}s = c\ \mathrm{d}c.$$

It was the fundamental equation of motion for Boscovich, serving to explain all the dynamic relationships easily in terms of the integral calculus (*admodum facile ope calculi integralis*). From the letter to Federico Sanvitali, it is known how, «passing to the integral», Boscovich explained free fall but neither at that time nor now did he show how this equation could explain a motion caused by an arbitrary given law of forces.

Concerning the choice of method, Boscovich referred to his previously known attitude that the solution could be obtained both in terms of the integral calculus and in terms of geometrical constructions, but the geometrical expression was more acceptable, saying: «Really, omitting the integral calculus, I shall expose here entirely expeditious geometrical constructions that perfect the matter and are more familiar

[26] *Ibidem*, pp. 15-16, n. 18.

[27] R.J. Boscovich, *De compositione motuum, ac virium agentium secundum eandem rectam*, in [1755a] *Supplementum ad librum primum*, nn. 149-195: 149, pp. 374-380: 374.

[28] On dependence of the law of forces on distance and time see R.J. Boscovich, *De compositione motuum* ..., cit., in [1755a] nn. 150, 165; on its representation with the help of scale see *ibidem*, n. 149.

to people» [29]. Having chosen the geometrical approach to the foundations of dynamics, he carried out the analysis of the dynamic diagram (see Figure 3), which was thematically identical to the researched passage from the treatise *De viribus vivis*. Therefore, it is instructive to compare these two texts in order to establish if, how, and why the improvement or simply alteration, relating to the preceding treatise, appears in the supplement *De compositione motuum*.

When he decided to systematically expose the basic dynamic relations Boscovich eliminated a number of revealed imprecisions. In the supplement he consistently used the notion *vis acceleratrix*, where the acceleration was not yet recognized, but just the force. Infinitely small quantities of time and velocity are called *infinitesimal* ones. In all four coordinate systems (see Figures 5b-8b) the area under the curve is understood as the sum of infinitesimal rectangles. On the basis of such a summation its meaning in mechanics is determined (cf. nn. 152, 154, 160, 163 *infra*). Further, the physical interpretation for proportion of these areas is given in keeping with the Euclidian ideal (cf. nn. 153, 155, 161, 164 *infra*).

It is supposed that the starting curve $f(t)$ is continuous (see Figure 5b), thereby the continuity of time and of the corresponding force is expressed by the idea of flux. The time interval is created by the continuous *duration* of a moment, and the curve $f(t)$ is traced by the continuous *motion* of the extreme point of an ordinate, used for expressing the momentary force [30]. The precision of the initial assumption follows from Boscovich's intensive study of continuity in geometry and nature in 1754, crowned with publishing the treatise *De continuitatis lege* that immediately preceded the final editing of his supplements to the first volume of Stay's poem.

Cherishing of the geometrical approach is particularly seen in two characteristic details. The first one is the geometrical proof of the general theorem of infinitesimal geometry concerning the increment of the square, as Boscovich called the statement:

$$d(x^2) = 2x dx$$ [31].

If a quantity is described by the line AH, then the square of this quantity (in the algebraic sense) is expressed by the square AHIB (see Figure 9). If the infinitesimal increment of quantity is expressed by the small line Hh, the *gnomon* HIBbihH presents the corresponding increment of the square of the considered quantity. Each of the equal figures IHhi, IBbi that form the gnomon is obtained, in accordance with Newton's method of fluents, by conducting the increasing side AH (= HI) → Ah along the increment Hh. Hence if AH = x, Hh = dx, then the area of gnomon is equal to $2x dx$, as Boscovich asserted in the statement of his theorem. Could a theorem of the same kind be stated for higher powers and

[29] «Verum, ipso integrali calculo omisso, exhibebo hic admodum expeditas geometricas constructiones, quae rem perficiunt, et vulgo notiores sunt» (R.J. Boscovich, *De compositione motuum*..., cit., in [1755a] pp. 374-375, n. 150).

[30] *Ibidem*, n. 151.

[31] *Ibidem*, nn. 158-159.

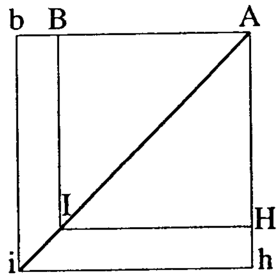

Fig. 9 - A gnomon as the geometrical explanation of the infinitesimal increment of a square. From Boscovich, *De compositione motuum*, in [1755a], Fig. 3.

in a general case? What is the range of the geometrical expression of the differential of a power? Already above the third power there is no possibility of visualization. Consequently, it cannot be accepted by the Euclidianly-educated audience. Therefore, there is no reason for the choice Boscovich made regarding the mathematical method.

The second detail is the geometrical expression of the reciprocal value that is directly connected with the definition of the function $\frac{1}{v}(s)$ (see Figure 8b). In keeping with Euclid's *Elements* VI, prop. 17, Boscovich established the equality between the rectangle with changeable sides and the constant, obviously unit-based square:

$$HM \cdot HL = AR^2, \ AR = 1,$$

where HM represents the velocity attained in time AH and HL represents the reciprocal value of velocity, the distance covered being AH. Further, Boscovich explained what occurs if the initial velocity is equal to zero, examining the procedure of tending towards the very beginning of motion:

$$HM \to 0$$

$$\Rightarrow HL \to \infty$$

$\Rightarrow$ FAHLO is the *asymptotic* area,

moreover it is finite since it represents the time spent on path AH [32]. Although Boscovich did not prove here the finiteness of the area with the help of mathematical apparatus, he at least discussed and postulated the character of the area considered.

Geometrical constructions, given in four coordinate systems (see Figures 5b-8b), enable one, according to Boscovich, to research any accelerated or decelerated motion, that is to establish the mutual relationships of velocity, distance and time for the given law of forces, whether it has been given in the form of $f(t)$

[32] *Ibidem*, n. 163.

or *f*(*s*) [33]. With this general attitude Boscovich explicitly determined the domain of validity and study of fundamental relations in dynamics and supported it by a range (number) of particular cases: uniformly accelerated motion (cf. nn. 166-168 *infra*), uniform motion (cf. n. 169 *infra*), non-uniformly accelerated motion (cf. nn. 172-175 *infra*), motion generated by the action of central force proportional to the distance from the centre (cf. nn. 177-185 *infra*), and motion generated by the action of central force in inverse proportion to the square of distance from the centre (cf. nn. 186-190 *infra*). He made this assertion for the first time in his treatise *De viribus vivis*: «From such a nature of curves [in the diagram] it is easy in foundations of mechanics to derive everything pertinent to these accelerated or decelerated motions» [34], while in the supplement *De compositione motuum* he deepened it with the systematic exposition and the consistent use of geometrical expressions of the linear and superficial infinitesimal, as well as with applications to various laws of forces.

Next Contexts: Theory of Forces and Fluid Mechanics

Finally, Boscovich's foundation of dynamics, formed in such a way, appears in time in new contexts. In his master-piece *Theoria philosophiae naturalis*, at the very beginning of his exposition of mechanics, Boscovich used his explanation of motion examining «the curve of forces that all the phenomena of motion depend on» (*quae pertinent ad ipsam virium curvam, a qua utique motuum phaenomena pendent omnia*) [35]. Once again he here established that the area under the curve *f*(*s*) represents the increment or decrement of the square of the velocity, but this time he said that it is in the same way easy to prove that statement by means of geometry (*itidem ope Geometriae demonstratur facile*), and even easier by means of integral calculus (*multo facilius ... ope calculi integralis*) [36].

By saying so, Boscovich dealt with the examination of relations between the force u, velocity c, time t, and distance s, which is reduced to the fundamental equation $c\, dc = u\, ds$. Hence he drew a conclusion: in any finite interval of time, the increase or decrease of the square of the velocity is represented by the area corresponding to that part of the axis which represents the distance covered. Hence it follows that, if through any distance the force on the point remains constant, but the moving body arrives at the beginning of this distance with any velocity, then the difference between the square of the final velocity and the square of initial velocity will always be the same. There exists a family of graphs, i.e.a. diagrammatic representation of the velocities wich are generated or destroyed as a particle moves, depending on whether its motion is in the same direction as the direction of the force, or in the opposite direction [37].

[33] «Invenientur nimirum relationes velocitatum, spatiorum, temporum inter se, data virium lege, sive ea a temporibus pendeat, sive a spatiis» (*ibidem*, p. 376, n. 165).

[34] «Ex ejusmodi autem linearum natura, omnia quae ad hosce motus acceleratos, aut retardatos pertinent in elementis Mechanicae facile deducuntur» (R.J. Boscovich [1745a], p. 19, n. 23).

[35] R.J. Boscovich [1763], n. 166, p. 77.

[36] *Ibidem*, p. 81, n. 176.

[37] «Ut nimirum habeatur scala quaedam velocitatum, quae in accessu puncti cujusvis ad aliud punctum, vel recessu generantur, vel eliduntur» (*ibidem*, p. 81, n. 176).

Boscovich applied these conclusions to describe the motion of a particle not situated on one of the intersection points of his curve of forces. A particle will continue to move towards the next intersection point of Boscovich's curve of forces in its direction of motion, with an accelerated motion according to the 'law' described above [38].

In *Theoria* Boscovich also investigated asymptotic areas in accordance with his methodological choice. An asymptotic area can be infinite or a finite one of any value, which can be proved, according to Boscovich, geometrically (*etiam geometrice*), but more easily by the elementary integral calculus (*sed multo facilius demonstratur calculi integrali admodum elementari*) [39]. For the general hyperbola we have:

$$x^m y^n = 1, \; y = x^{-m/n},$$

$$\int x^{-m/n} \, dx = \frac{n}{n-m} x^{-m/n} x + A = \frac{n}{n-m} xy + A.$$

For $n-m>0$ or $\frac{m}{n}<1$ the area is finite, but for $n-m=0$ or $\frac{m}{n}=1$, and a fortiori for $n-m<0$ or $\frac{m}{n}>1$, the area is infinite.

This mathematical conclusion of Boscovich's should be compared with his previously postulated physical condition regarding the asymptotic area: «in order that these forces [considered in the dependence on the distance] could destroy each other, however great the velocity, the area must be greater than any finite one» [40]. Therefore it means, although I have found it nowhere in Boscovich's text, that the asymptotic arc of Boscovich's curve that describes the generation or disappearance of velocity has a form: $y=x^{-m/n}$, where $\frac{m}{n}>1$. This is also an example of how Boscovich knew to condition an open problem in geometry or infinitesimal analysis due to the character of investigated physical event.

[38] «Usque ad distantiam limitis proximi, motu semper accelerato, juxta legem expositam num. 176» (*ibidem*, p. 86, n. 190).

[39] *Ibidem*, p. 80, n. 175; see the proof of Boscovich's statement in annotation (l), pp. 80-81.

[40] *Ibidem*, a. (f), p. 53, n. 118: «quare ut illae vires sint pares extinguendae velocitati cuivis utcunque magnae, debet illa area esse omni finita major». Cf. Boscovich's earlier conclusion in Boscovich [1754a], p. 76; n. 164: «Cumque idem discursus redeat pro quavis utcumque magna velocitatum differentia, patet, ejusmodi debere esse vim repulsivam, ut corpora nunquam ad immediatum contactum deveniant, et ipsa sit par extinguendae velocitati utcumque magnae, adeoque ut imminutis in infinitum distantiis excrescat ultra quoscumque limites in infinitum, atque ita excrescat, ut recta ipsi proportionalis ducta in rectam exprimentem distantias describat aream infinitam (cum nimirum illi areae in Mechanica demonstretur proportionale quadratum velocitatis genitae, vel destructae) ad quam areae infinitatem requiritur, ut vis decrescat non minus, quam in simplici ratione distantiae; nam ea ratio exhibet Hyperbolam Conicam inter asymptotos habentem aream infinitam, et in omni Hyperbolarum familia eae omnes, quae ordinatas habent minus crescentes, habent aream finitam, quae habent magis crescentes, habent ipsam aream itidem infinities magis infinitam». For a detailed interpretation of this passage see E. Stipanić, *Naučni i istorijski komentar*, in R.J. Boscovich [1975], pp. 95-158: 156-157.

In the letter to Giovanni Arnolfini of 2 February 1770 Boscovich returned to the explanation of the dynamic diagram (see Figure 4). It is unknown whether or not Arnolfini's attitude towards the infinitesimal calculus influenced that step. From the four coordinate systems, he chose in the letter the two which show the law of force in the dependence on time and space and serve for the geometrical presentation of velocity and velocity squared [41]. This is to say that the area under the curve $f(t)$ represents velocity, while the area under the curve $f(s)$ represents the velocity squared. In Boscovich's opinion «these theorems are the foundation of all of mechanics», (*questi teoremi sono il fondamento di tutta la Mecanica*) [42], and examples of the forces he stated are gravity and the elastic force. Obviously, while writing the letter, Boscovich relied on the original explanation of the diagram in his early treatise *De viribus vivis*.

Further, in the same letter, Boscovich elucidated his already mentioned consideration which leads to the equation $c\,\mathrm{d}c = f\,\mathrm{d}s$, and finally used the integral calculus to obtain the expression for velocity squared. In 1770 he explicitly started from the formulae:

$$c = ft, \quad s = ct$$ [43].

It means that Boscovich, at his late age, did not distinguish between the force as an action of cause (*la forza, l'azione della potenza, la pressione generatrice della velocità*) [44] and the acceleration as an immediate effect of such an action of the force. This undifferentiated conception about force is also found in the treatise *De viribus vivis*.

Conclusion

The problem of *generatio velocitatis* and its genesis reflected Boscovich's relation to the choice of mathematical method in the founding of dynamics in the period 1745-1770. In the entire work of Roger Boscovich it is an exceptional case that does not allow simplifications. Based on investigated sources it is possible to recognize several phases in Boscovich's search as far as the mathematical method is concerned:

(a) the infinitesimal analysis of the implicit equation of the parabola, when Boscovich, introducing the notion of the increment of an algebraic quantity, used the method of differentiation and integration as a numerical procedure and called the problem purely geometrical;

(b) the dynamic analysis of free fall, when Boscovich, introducing the corresponding dynamic quantities, reduced the previous result to the fundamental equation of motion $f\,\mathrm{d}s = u\,\mathrm{d}u$ and estimated different starting conditions of the free fall by integration;

[41] R.J. Boscovich [1963a], pp. 13-14.

[42] *Ibidem*, 14.

[43] *Ibidem*, 16.

[44] *Ibidem*, 25.

(c) the analysis of s'Gravesande's dynamic diagram as a generalisation of free fall, when Boscovich used geometrical ideas of the linear and superficial infinitesimals in their defining forms, and thereby the fundamental idea of infinitesimal calculus that one can reach the general solution on the basis of local properties of functions;
(d) systematic exposition of the problem of *generatio velocitatis* by means of geometrical constructions coupled with strong stress on the Euclidean ideal: the equality of the rectangle with variable sides and the constant square as a way of determining the reciprocal value of a function, proportions of the areas under the curve as a way of establishing the physical meaning of the definite integral, the gnomon as the geometrical description of the infinitesimal increment of the square of a varying quantity;
(e) application of Boscovich's continuous curve of forces, when Boscovich used the elementary integral calculus to explain the area under the curve including asymptotic area;
(f) explanation of the generation of velocity as the foundation of the whole of mechanics and, accordingly, hydrostatics, in both the geometrical and infinitesimal forms, when Boscovich repeated his general solution in the original rudimentary form from *De viribus vivis*.

In the general form as well as in applications to particular cases Boscovich's conclusions do not contain novelties in their explanations of relations between the basic dynamic quantities. Moreover, they are not at the level of the then current understanding considering the fact that Boscovich here does not distinguish the force from the acceleration, i.e., he does not use the notion of acceleration as a physical quantity. On the contrary, this cognitive process unfolds the reasons for Boscovich's fundamental choice of mathematical method: the geometrical method.

Although he was acquainted with the fundamentals of the infinitesimal calculus, Boscovich kept to the level of basic ideas of the calculus, not having used its numerical effectivness. He persevered in his own Euclidean ideal and he used the infinitesimal in its defining form in his geometrical constructions. The infinitesimal, geometrically presented and connected with «the transition to an integral», offered him a guarantee he had found the general solution. Is such an understanding of the infinitesimal sufficient for developing the infinitesimal geometry he intended to work out, having announced it publicly many times, in the fourth, never completed volume of *Elementa universae matheseos*? According to the announcement in the foreword to the third volume of his *Elementa*, Boscovich wanted to prove the fundamentals of the infinitesimal calculus with geometrical rigor (*geometrico rigore*), and then research the general properties of curves (cusps, points of inflection, infinite branches, osculating circles, evolutes, maxima and minima) [45]. If by 'geometrical rigor' he meant the procedure by which he showed what $d(x^2)$ was equal to in his supplement *De compositione motuum* to Stay's philosophy in verses, and if he, in that way, wanted to elaborate the whole of differential geometry, it is clear he had to meet insurmountable difficulties, and

[45] R.J. Boscovich, *Auctoris praefatio*, in *Idem* [1754b], vol. III, pp. III-XXVI: XXV.

finally gave up the idea of writing the fourth volume. However, it did not prevent him from using the fundamentals of the infinitesimal calculus with «geometrical rigor» to shape his own ideas.

Most scholars call Boscovich's approach to the infinitesimal calculus a failure and consider it to be the main reason for the inappropriate acceptance of Boscovich's results in the 18th and 19th centuries, especially if Boscovich's contributions to theoretical astronomy and the explanation of the structure of matter are considered. Although this is true, at the same time it is important also to mention Boscovich's ingenuity. It is shown in the fact that even such ideas of his as these, although not dominant ones in the period, were fertilized and incorporated into his theory of forces in nature, as is undoubtedly seen in the analysed passage from *Theoria philosophiae naturalis*.